

Using U spin to extract γ from charmless $B \rightarrow PPP$ decays

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ABSTRACT: Some years ago, a method was proposed for measuring the CP-violating phase γ using pairs of two-body decays that are related by U-spin reflection ($d \leftrightarrow s$). In this paper we adapt this method to charmless $B \rightarrow PPP$ decays. Time-dependent Dalitz-plot analyses of these three-body decays are required for the measurement of the mixing-induced CP asymmetries. However, isobar analyses of the decay amplitudes are not necessary. A potential advantage of using three-body decays is that the effects of U-spin breaking may be reduced by averaging over the Dalitz plot. This can be tested independently using the measurements of direct CP asymmetries and branching ratios in three-body charged B decays.

KEYWORDS: B-Physics, CP violation

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1 Introduction

The standard model (SM) explanation of CP violation is that it is due to a phase in the Cabibbo-Kobayashi-Maskawa (CKM) quark mixing matrix. This phase information is elegantly encoded in the unitarity triangle, whose interior CP-violating angles are α , β and γ [1]. Using B decays, a great deal of effort has gone into measuring these angles in many different ways, along with the sides of the unitarity triangle, to search for inconsistencies that would indicate the presence of new physics (NP). Unfortunately, to date no such indications have been seen. This suggests that the NP is more massive than hoped for (which is consistent with the absence of NP signals at the LHC), and that the observation of its effects on CP violation in the B system will require measurements of greater precision.

One interesting procedure for searching for NP involves the CKM phase γ . The conventional way of measuring γ uses the tree-level decay $B^- \rightarrow D^{(*)} K^{(*)-}$ [2–6]. Its latest value is $\gamma = (71.7_{-7.4}^{+7.1})^\circ$ [7].¹ However, suppose that γ could be measured using decays that have significant (gluonic or electroweak) penguin contributions. If NP is present, it is likely to affect the (loop-level) penguins, in which case the extracted value of γ would be different from that found using $B^- \rightarrow D^{(*)} K^{(*)-}$. That is, one can probe NP by comparing the “tree-level” and “loop-level” values of γ . (But note that, if there is NP, the “loop-level” value of γ will not be constant. It will generally vary, depending on which decays are used for its extraction.)

One example of this involves $B \rightarrow \pi K$ decays. (In what follows, we briefly describe the method, but we refer the reader to ref. [8] for full details.) There are four such decays:

¹Updated results and plots available at <http://ckmfitter.in2p3.fr/>.

$B^+ \rightarrow \pi^+ K^0$, $B^+ \rightarrow \pi^0 K^+$, $B_d^0 \rightarrow \pi^- K^+$ and $B_d^0 \rightarrow \pi^0 K^0$. Using these processes, one can measure nine observables: four branching ratios, four direct CP asymmetries, and one indirect (mixing-induced) CP asymmetry. However, assuming flavor SU(3) symmetry, the amplitudes can be written in terms of eight theoretical parameters: the magnitudes of the diagrams P'_{tc} , T' , C' , P'_{uc} , three relative strong phases, and the weak phase γ . (The value of the weak phase β is taken from the measurement of indirect CP violation in $B_d^0(t) \rightarrow J/\Psi K_S$ [7].) With more observables than theoretical parameters, one can perform a fit to extract γ . The value found is $\gamma = (35.3 \pm 7.1)^\circ$ [8], which differs from the tree-level value of γ by 3.5σ . While this is intriguing, one must remember that there is also an unknown theoretical uncertainty due to SU(3) breaking. Before any conclusions can be drawn, there must be other, independent determinations of loop-level values of γ .

In 1999, R. Fleischer proposed a method for extracting γ from $B_s^0 \rightarrow K^+ K^-$ and $B_d^0 \rightarrow \pi^+ \pi^-$, two decays whose amplitudes are related by U-spin ($d \leftrightarrow s$) symmetry [9]. Since penguin contributions are important for such decays, this method would determine a loop-level value of γ . It requires the measurement of the branching ratios and CP asymmetries, both direct and indirect, of both decays. This method is unaffected by final-state interactions; its theoretical accuracy is limited only by the size of U-spin-breaking effects. The factorizable U-spin-breaking corrections are calculable theoretically in terms of form factors and decay constants [9–11]. However, the precise value of the nonfactorizable U-spin-breaking correction is unknown, though it may be sizeable [12].

Recently, the direct and indirect CP asymmetries in $B_s^0 \rightarrow K^+ K^-$ were measured by the LHCb Collaboration [13], and they carried out the above extraction of γ [14]. Allowing for a U-spin-breaking error of 50%, they find $\gamma = (63.5_{-6.7}^{+7.2})^\circ$. However, if the theoretical error is $\geq 60\%$, the uncertainty on γ is much larger.

It was pointed out in ref. [9] that, with an additional dynamical assumption, one could replace $B_s^0 \rightarrow K^+ K^-$ with $B_d^0 \rightarrow \pi^\mp K^\pm$, and analyses with this second decay were carried out in refs. [10, 11, 15]. However, ref. [16] finds that the experimental data suggest that there may be a large nonfactorizable U-spin-breaking correction between $B_d^0 \rightarrow \pi^\mp K^\pm$ and $B_d^0 \rightarrow \pi^+ \pi^-$. This would lead to a large (unknown) theoretical error in the extraction of γ using $B_d^0 \rightarrow \pi^\mp K^\pm$ and $B_d^0 \rightarrow \pi^+ \pi^-$.

The main purpose of the present paper is to note that the method of ref. [9] can also be applied to charmless $B \rightarrow PPP$ decays (P is a pseudoscalar meson) whose amplitudes are related by U spin. The key point is that, by using the Dalitz plots of the three-body decays, the effect of U-spin breaking may be greatly reduced. If this is possible — and there is an independent test to see if the procedure works — then the loop-level value of γ can be determined with little theoretical error. This will then provide a clean test for NP.

Note that, under flavor SU(3) symmetry, the three final-state particles in charmless $B \rightarrow PPP$ decays are treated as identical, so that the six permutations of these particles must be considered. There have been a number of papers recently that use the fully-symmetric final state [17–21], which can be obtained using an isobar analysis of the Dalitz plot. However, we stress that such an analysis is *not* needed for the above method of extracting γ — the full Dalitz plot is used.

Examples of pairs of decays to which this method can be applied are (i) $B_s^0 \rightarrow K_S \pi^+ \pi^-$ ($\bar{b} \rightarrow \bar{d}$) and $B_d^0 \rightarrow K_S K^+ K^-$ ($\bar{b} \rightarrow \bar{s}$), and (ii) $B_s^0 \rightarrow K_S K^+ K^-$ ($\bar{b} \rightarrow \bar{d}$) and $B_d^0 \rightarrow$

$K_S\pi^+\pi^-$ ($\bar{b} \rightarrow \bar{s}$). The time-dependent Dalitz plots for $B_d^0 \rightarrow K_SK^+K^-$ and $B_d^0 \rightarrow K_S\pi^+\pi^-$ were measured by the BaBar and Belle Collaborations [22–25], and a study of $B_{(s)}^0 \rightarrow K_S h^+ h'^-$ was made by the LHCb Collaboration [26]. For the B_s^0 decays, it appears that $B_s^0 \rightarrow K_S\pi^+\pi^-$ is more promising experimentally. The first observation of this decay was reported in ref. [26], and a study of the future prospects for the measurement of its time-dependent Dalitz plot was presented in ref. [27]. Hopefully the method will be applied to decays $B_s^0 \rightarrow K_S\pi^+\pi^-$ and $B_d^0 \rightarrow K_SK^+K^-$ to extract γ .

In section 2, we briefly discuss Dalitz plots and the distinction between the final states f and \bar{f} . The U-spin relation between $\bar{b} \rightarrow \bar{d}$ and $\bar{b} \rightarrow \bar{s}$ decays is discussed in section 3. In section 4, we present the method for extracting γ from a Dalitz-plot analysis of three-body decays. The subject of U-spin-breaking effects — the theoretical idea of how they may be reduced in three-body decays, and experimental tests of this hypothesis — is examined in section 5. We conclude in section 6.

2 Dalitz plots

Three-body B decays are usually described using a Dalitz plot. Consider the decay $B \rightarrow P_1 P_2 P_3$, in which each pseudoscalar P_i has momenta p_i . One can construct the three Mandelstam variables $s_{ij} \equiv (p_i + p_j)^2$, where p_i is the momentum of each P_i . These are not independent, but obey $s_{12} + s_{13} + s_{23} = m_B^2 + m_1^2 + m_2^2 + m_3^2$. The $B \rightarrow P_1 P_2 P_3$ Dalitz plot is a measure of the decay rate as a function of two Mandelstam variables.

In the present paper we focus on the decays $B_{d,s}^0 \rightarrow K_S(p_1)h^+(p_2)h^-(p_3)$ ($h = K, \pi$). At the quark level, the final states $f = K_S\pi^+\pi^-$ and $K_SK^+K^-$ are self-conjugate. However, when the momenta are considered, one has $\bar{f} \neq f$. The point is that the CP conjugate of $f = K_S(p_1)h^+(p_2)h^-(p_3)$ is $\bar{f} = K_S(\bar{p}_1)h^-(\bar{p}_2)h^+(\bar{p}_3)$, where \bar{p}_i is p_i with the direction of the three-momentum reversed. Note that reversing the direction of the three momenta does not affect the Mandelstam variables, since $s_{ij} = (p_i + p_j)^2 = (\bar{p}_i + \bar{p}_j)^2 = \bar{s}_{ij}$. Thus, in this case the difference between f and \bar{f} arises from an exchange of the indices 2 and 3.

The distinction between f and \bar{f} must be kept in mind throughout the paper. Because f is self-conjugate at the quark level, both B^0 and \bar{B}^0 can decay to it, and similarly for \bar{f} . Now, at different points in the analysis we consider the direct CP asymmetry. However, because $\bar{f} \neq f$, there are two of these. One compares $B^0 \rightarrow f$ and $\bar{B}^0 \rightarrow \bar{f}$ decays, the other $B^0 \rightarrow \bar{f}$ and $\bar{B}^0 \rightarrow f$. Things are similar for the indirect CP asymmetry, which arises because both B^0 and \bar{B}^0 can decay to the same final state. Thus, one indirect asymmetry involves the interference of the amplitudes for $B^0 \rightarrow f$ and $\bar{B}^0 \rightarrow f$, while the other involves the interference of $\mathcal{A}(B^0 \rightarrow \bar{f})$ and $\mathcal{A}(\bar{B}^0 \rightarrow \bar{f})$.

3 U-spin relation

In this section we discuss the U-spin relation that is central to our method for extracting γ . We begin by reviewing the relation for two-body decays.

3.1 Two-body decays

Consider a pair of $B \rightarrow PP$ decays whose amplitudes are related by U-spin reflection ($d \leftrightarrow s$). (This discussion follows ref. [28].) One is a $\bar{b} \rightarrow \bar{d}$ decay, the other $\bar{b} \rightarrow \bar{s}$. There are five such pairs [16]: $(B_d^0 \rightarrow \pi^+\pi^-, B_s^0 \rightarrow K^+K^-)$, $(B_s^0 \rightarrow \pi^+K^-, B_d^0 \rightarrow \pi^-K^+)$, $(B^+ \rightarrow K^+\bar{K}^0, B^+ \rightarrow \pi^+K^0)$, $(B_d^0 \rightarrow K^0\bar{K}^0, B_s^0 \rightarrow \bar{K}^0K^0)$, $(B_d^0 \rightarrow K^+K^-, B_s^0 \rightarrow \pi^+\pi^-)$.

The $\bar{b} \rightarrow \bar{d}$ amplitude can be written

$$\begin{aligned}\mathcal{A}_d &= A_u V_{ub}^* V_{ud} + A_c V_{cb}^* V_{cd} + A_t V_{tb}^* V_{td} \\ &= (A_u - A_t) V_{ub}^* V_{ud} + (A_c - A_t) V_{cb}^* V_{cd} \\ &\equiv V_{ub}^* V_{ud} T_d + V_{cb}^* V_{cd} P_d.\end{aligned}\tag{3.1}$$

In the above, the A_i each represent a linear combination of diagrams, and we have used the unitarity of the CKM matrix ($V_{ub}^* V_{ud} + V_{cb}^* V_{cd} + V_{tb}^* V_{td} = 0$) to write the second line. T_d and P_d are simply the quantities that are multiplied by the given CKM matrix elements — they do not represent individual “tree” and “penguin” diagrams. The $\bar{b} \rightarrow \bar{s}$ amplitude can be written similarly:

$$\mathcal{A}_s = V_{ub}^* V_{us} T_s + V_{cb}^* V_{cs} P_s.\tag{3.2}$$

The CP-conjugate amplitudes $\bar{\mathcal{A}}_d$ and $\bar{\mathcal{A}}_s$ are obtained from the above by changing the signs of the weak phases:

$$\bar{\mathcal{A}}_d = V_{ub} V_{ud}^* T_d + V_{cb} V_{cd}^* P_d, \quad \bar{\mathcal{A}}_s = V_{ub} V_{us}^* T_s + V_{cb} V_{cs}^* P_s.\tag{3.3}$$

We then have

$$\begin{aligned}|\mathcal{A}_d|^2 - |\bar{\mathcal{A}}_d|^2 &= 4 \operatorname{Im}(V_{ub}^* V_{ud} V_{cb} V_{cd}^*) \operatorname{Im}(T_d P_d^*), \\ |\mathcal{A}_s|^2 - |\bar{\mathcal{A}}_s|^2 &= 4 \operatorname{Im}(V_{ub}^* V_{us} V_{cb} V_{cs}^*) \operatorname{Im}(T_s P_s^*).\end{aligned}\tag{3.4}$$

Now, the unitarity of the CKM matrix implies [29]

$$\operatorname{Im}(V_{ub}^* V_{us} V_{cb} V_{cs}^*) = -\operatorname{Im}(V_{ub}^* V_{ud} V_{cb} V_{cd}^*),\tag{3.5}$$

and in the U-spin limit we have

$$T_d = T_s, \quad P_d = P_s.\tag{3.6}$$

U-spin symmetry therefore leads to a relation between the $\bar{b} \rightarrow \bar{d}$ and $\bar{b} \rightarrow \bar{s}$ decays:

$$|\mathcal{A}_d|^2 - |\bar{\mathcal{A}}_d|^2 = -[|\mathcal{A}_s|^2 - |\bar{\mathcal{A}}_s|^2].\tag{3.7}$$

In general, there are four observables in the $\bar{b} \rightarrow \bar{d}$ and $\bar{b} \rightarrow \bar{s}$ processes: the branching ratios B_d and B_s , and the direct CP asymmetries A_d^{CP} and A_s^{CP} . Eq. (3.7) implies that these are not independent, but obey [9, 28]

$$-\frac{A_s^{CP}}{A_d^{CP}} \frac{\tau(B_d^0) B_s}{\tau(B_s^0) B_d} = 1.\tag{3.8}$$

Thus, there are only three independent observables.

3.2 $B_{d,s}^0 \rightarrow K_S h^+ h^-$ decays

We now turn to $B_{d,s}^0 \rightarrow K_S h^+ h^-$ decays. For definitiveness, we focus on the pair ($B_s^0 \rightarrow K_S \pi^+ \pi^-$ ($\bar{b} \rightarrow \bar{d}$), $B_d^0 \rightarrow K_S K^+ K^-$ ($\bar{b} \rightarrow \bar{s}$)), but the results can be equally applied to ($B_s^0 \rightarrow K_S K^+ K^-$ ($\bar{b} \rightarrow \bar{d}$), $B_d^0 \rightarrow K_S \pi^+ \pi^-$ ($\bar{b} \rightarrow \bar{s}$)).

As discussed in section 2, one must pay attention to the momenta of the final-state particles. Let us define $f_d \equiv K_S(p_1)\pi^+(p_2)\pi^-(p_3)$ and $\bar{f}_d \equiv K_S(p_1)\pi^+(p_3)\pi^-(p_2)$, and similarly for f_s and \bar{f}_s . Now consider $\mathcal{A}_d = \mathcal{A}(B_s^0 \rightarrow f_d)$ and $\mathcal{A}_s = \mathcal{A}(B_d^0 \rightarrow f_s)$. The decay amplitudes \mathcal{A}_d and \mathcal{A}_s are again given by eqs. (3.1) and (3.2), respectively, and are repeated below for convenience:

$$\mathcal{A}_d = V_{ub}^* V_{ud} T_d + V_{cb}^* V_{cd} P_d, \quad \mathcal{A}_s = V_{ub}^* V_{us} T_s + V_{cb}^* V_{cs} P_s. \quad (3.9)$$

As these are three-body decays, $T_{d,s}$ and $P_{d,s}$ are all momentum-dependent. This means that T_d takes different values at different points of the Dalitz plot, and similarly for T_s and $P_{d,s}$. For the CP-conjugate amplitudes, we have

$$\bar{\mathcal{A}}_d = V_{ub} V_{ud}^* \bar{T}_d + V_{cb} V_{cd}^* \bar{P}_d, \quad \bar{\mathcal{A}}_s = V_{ub} V_{us}^* \bar{T}_s + V_{cb} V_{cs}^* \bar{P}_s. \quad (3.10)$$

Because the final states in the CP-conjugate decays are not the same as in the decays (p_2 and p_3 are exchanged), $T_d \neq \bar{T}_d$, and similarly for T_s and $P_{d,s}$.

We then have [30]

$$\begin{aligned} |\mathcal{A}_d|^2 - |\bar{\mathcal{A}}_d|^2 &= 2 \operatorname{Im}(V_{ub}^* V_{ud} V_{cb} V_{cd}^*) \operatorname{Im}(T_d P_d^* + \bar{T}_d^* \bar{P}_d), \\ |\mathcal{A}_s|^2 - |\bar{\mathcal{A}}_s|^2 &= 2 \operatorname{Im}(V_{ub}^* V_{us} V_{cb} V_{cs}^*) \operatorname{Im}(T_s P_s^* + \bar{T}_s^* \bar{P}_s). \end{aligned} \quad (3.11)$$

In the U-spin limit we have $T_d = T_s$, $P_d = P_s$, $\bar{T}_d = \bar{T}_s$, $\bar{P}_d = \bar{P}_s$, and the U-spin relation of eq. (3.7) is reproduced. However, since the amplitudes themselves are now momentum dependent, this relation holds at each point in the Dalitz plots.

As in the two-body case, the U-spin relation implies a relation among the observables, similar to eq. (3.8). This relation involves $B^0 \rightarrow f$ and $\bar{B}^0 \rightarrow \bar{f}$ decays, and can be written as

$$-\frac{a_s^{CP}}{a_d^{CP}} \frac{\tau(B_d^0) b_s}{\tau(B_s^0) b_d} = 1. \quad (3.12)$$

Here, a_q^{CP} and b_q are, respectively, the direct CP asymmetry and branching ratio defined locally, i.e., at a particular Dalitz-plot point. They are both momentum-dependent quantities.

The analysis can be repeated for the case where $\mathcal{A}_d = \mathcal{A}(B_s^0 \rightarrow \bar{f}_d)$ and $\mathcal{A}_s = \mathcal{A}(B_d^0 \rightarrow \bar{f}_s)$. Here we have

$$\mathcal{A}_d = V_{ub}^* V_{ud} \bar{T}_d + V_{cb}^* V_{cd} \bar{P}_d, \quad \mathcal{A}_s = V_{ub}^* V_{us} \bar{T}_s + V_{cb}^* V_{cs} \bar{P}_s. \quad (3.13)$$

and

$$\bar{\mathcal{A}}_d = V_{ub} V_{ud}^* T_d + V_{cb} V_{cd}^* P_d, \quad \bar{\mathcal{A}}_s = V_{ub} V_{us}^* T_s + V_{cb} V_{cs}^* P_s. \quad (3.14)$$

Once again, the U-spin relation of eq. (3.7) is reproduced. And there is a relation like eq. (3.12) among the observables. This relation involves $B^0 \rightarrow \bar{f}$ and $\bar{B}^0 \rightarrow f$ decays.

The point here is that, for three-body decays, there are two U-spin relations among the observables. These involve the same momentum-dependent hadronic parameters.

4 Extraction of γ

Here we present the details of how γ can be extracted from a U-spin analysis of $B_{d,s}^0 \rightarrow K_S h^+ h^-$ decays. We begin with a review of the method for two-body decays.

4.1 Two-body decays

The method proposed by Fleischer for extracting γ from $B_s^0 \rightarrow K^+ K^-$ and $B_d^0 \rightarrow \pi^+ \pi^-$ [9] works as follows. The amplitude for the $\bar{b} \rightarrow \bar{d}$ decay ($B_d^0 \rightarrow \pi^+ \pi^-$) is given in eq. (3.1), which can be written

$$\mathcal{A}_d = |V_{ub}^* V_{ud}| e^{i\gamma} T_d - |V_{cb}^* V_{cd}| P_d, \quad (4.1)$$

where we have used $|V_{cd}| = -V_{cd}$. The amplitude for the $\bar{b} \rightarrow \bar{s}$ decay ($B_s^0 \rightarrow K^+ K^-$) can be written similarly:

$$\mathcal{A}_s = |V_{ub}^* V_{us}| e^{i\gamma} T_s + |V_{cb}^* V_{cs}| P_s. \quad (4.2)$$

In the U-spin limit, we have $T_d = T_s \equiv T$ and $P_d = P_s \equiv P$. Assuming that the magnitudes of the CKM matrix elements are known, \mathcal{A}_d and \mathcal{A}_s each contain the same four unknown parameters: $|T|$, $|P|$, their relative strong phase, and γ .

Above [eq. (3.8)], it was noted that the branching ratios and the direct CP asymmetries of these two decays are not independent. Thus, γ cannot be extracted from the measurements of these observables alone, since there are more unknown theoretical parameters (four) than observables (three). However, if the indirect CP asymmetries in both $B_d^0 \rightarrow \pi^+ \pi^-$ and $B_s^0 \rightarrow K^+ K^-$ are also measured, and values for the B_d^0 - \bar{B}_d^0 and B_s^0 - \bar{B}_s^0 mixing phases (β and β_s , respectively) are taken from independent measurements, there will be more observables (five) than unknowns, which will allow γ to be extracted.

4.2 $B_{d,s}^0 \rightarrow K_S h^+ h^-$ decays

A similar logic can be applied to three-body decays. However, care must be taken in identifying the observables to be used, and in establishing how these observables depend on the unknown theoretical parameters.

The point is the following. If a final state f is self-conjugate at the quark level, both B^0 and \bar{B}^0 can decay to it. In the case of two-body decays, the fact that f is self-conjugate implies that $\bar{f} = f$, so that the two decays $B^0, \bar{B}^0 \rightarrow f$ must be considered. However, as noted in section 2, for three-body decays, a self-conjugate f still has $\bar{f} \neq f$, since f and \bar{f} correspond to different points of the Dalitz plot. In this case, the analysis must consider the four decays $B^0, \bar{B}^0 \rightarrow f, \bar{f}$. The time dependence of two-body decays has been analyzed in refs. [31, 32]. Below we adapt this analysis to three-body decays.

In the presence of B^0 - \bar{B}^0 mixing, the B_L and B_H states (L is light, H is heavy) are mixtures of B^0 and \bar{B}^0 . The physical time-dependent neutral B -meson states can then be expressed as

$$\begin{aligned} |B_{\text{phys}}^0(t)\rangle &= f_+(t) |B^0\rangle + \frac{q}{p} f_-(t) |\bar{B}^0\rangle, \\ |\bar{B}_{\text{phys}}^0(t)\rangle &= \frac{p}{q} f_-(t) |B^0\rangle + f_+(t) |\bar{B}^0\rangle. \end{aligned} \quad (4.3)$$

Here $B_{\text{phys}}^0(t)$ ($\bar{B}_{\text{phys}}^0(t)$) is the state that is a B^0 (\bar{B}^0) at $t = 0$. In the above, $q/p = e^{-2i\phi_M}$, where ϕ_M is the weak phase of the mixing (the B_d^0 - \bar{B}_d^0 and B_s^0 - \bar{B}_s^0 mixing phases are β and β_s , respectively), and

$$f_+(t) = e^{-i(m-i\Gamma/2)t} \cos(\Delta\mu t/2), \quad f_-(t) = e^{-i(m-i\Gamma/2)t} i \sin(\Delta\mu t/2), \quad (4.4)$$

with

$$\begin{aligned} m &= (m_H + m_L)/2, & \Delta m &= m_H - m_L, \\ \Gamma &= (\Gamma_H + \Gamma_L)/2, & \Delta\Gamma &= \Gamma_H - \Gamma_L, \\ \Delta\mu &= \Delta m - i\Delta\Gamma/2. \end{aligned} \quad (4.5)$$

The decay amplitudes are then given by

$$\begin{aligned} \langle f|B_{\text{phys}}^0(t) \rangle &= \langle f|B^0 \rangle (f_+(t) + \lambda f_-(t)), \\ \langle \bar{f}|B_{\text{phys}}^0(t) \rangle &= \frac{q}{p} \langle \bar{f}|\bar{B}^0 \rangle (f_+(t)\bar{\lambda} + f_-(t)), \\ \langle f|\bar{B}_{\text{phys}}^0(t) \rangle &= \frac{p}{q} \langle f|B^0 \rangle (f_-(t) + \lambda f_+(t)), \\ \langle \bar{f}|\bar{B}_{\text{phys}}^0(t) \rangle &= \langle \bar{f}|\bar{B}^0 \rangle (f_-(t)\bar{\lambda} + f_+(t)), \end{aligned} \quad (4.6)$$

where

$$x \equiv \frac{\langle f|\bar{B}^0 \rangle}{\langle f|B^0 \rangle}, \quad \bar{x} \equiv \frac{\langle \bar{f}|B^0 \rangle}{\langle \bar{f}|\bar{B}^0 \rangle}, \quad \lambda \equiv \frac{q}{p}x, \quad \bar{\lambda} \equiv \frac{p}{q}\bar{x}. \quad (4.7)$$

In ref. [31] the assumption is made that $\Delta\Gamma = 0$. In ref. [32] it is noted that $\Delta\Gamma$ is nonzero in B_s^0 decays. Our expressions below therefore allow for a nonzero $\Delta\Gamma$.

The decay rates are proportional to the squares of the amplitudes, which take the form

$$\begin{aligned} |\mathcal{M}|^2(B_{\text{phys}}^0(t) \rightarrow f) &= \frac{1}{2}|A|^2 e^{-\Gamma t} \left[(1 - |x|^2) \cos(\Delta m t) + (1 + |x|^2) \cosh(\Delta\Gamma t/2) \right. \\ &\quad \left. - 2 \text{Im}(\lambda) \sin(\Delta m t) + 2 \text{Re}(\lambda) \sinh(\Delta\Gamma t/2) \right], \\ |\mathcal{M}|^2(\bar{B}_{\text{phys}}^0(t) \rightarrow f) &= \frac{1}{2}|A|^2 e^{-\Gamma t} \left[-(1 - |x|^2) \cos(\Delta m t) + (1 + |x|^2) \cosh(\Delta\Gamma t/2) \right. \\ &\quad \left. + 2 \text{Im}(\lambda) \sin(\Delta m t) + 2 \text{Re}(\lambda) \sinh(\Delta\Gamma t/2) \right], \\ |\mathcal{M}|^2(B_{\text{phys}}^0(t) \rightarrow \bar{f}) &= \frac{1}{2}|\bar{A}|^2 e^{-\Gamma t} \left[-(1 - |\bar{x}|^2) \cos(\Delta m t) + (1 + |\bar{x}|^2) \cosh(\Delta\Gamma t/2) \right. \\ &\quad \left. + 2 \text{Im}(\bar{\lambda}) \sin(\Delta m t) + 2 \text{Re}(\bar{\lambda}) \sinh(\Delta\Gamma t/2) \right], \\ |\mathcal{M}|^2(\bar{B}_{\text{phys}}^0(t) \rightarrow \bar{f}) &= \frac{1}{2}|\bar{A}|^2 e^{-\Gamma t} \left[(1 - |\bar{x}|^2) \cos(\Delta m t) + (1 + |\bar{x}|^2) \cosh(\Delta\Gamma t/2) \right. \\ &\quad \left. - 2 \text{Im}(\bar{\lambda}) \sin(\Delta m t) + 2 \text{Re}(\bar{\lambda}) \sinh(\Delta\Gamma t/2) \right], \end{aligned} \quad (4.8)$$

where $A \equiv \langle f|B^0 \rangle$, $\bar{A} \equiv \langle \bar{f}|\bar{B}^0 \rangle$, and we have used $|q/p| = 1$.

With the squares of the amplitudes in hand, we can now obtain expressions for the observables. Before doing so, there is one point that must be mentioned. Although we have referred to measurements at different points of the Dalitz plot, in practice it is only possible to make measurements in bins, i.e., over areas of the Dalitz plot centred at different points. The observables will then involve integrals over the Mandelstam variables representing these bins.

Using the first two equations of eq. (4.8), we can now construct the time-dependent CP-averaged rate and the CP asymmetry for the final state f :

$$\begin{aligned}\Gamma(t) &= \frac{1}{2}(\Gamma(B_{\text{phys}}^0(t) \rightarrow f) + \Gamma(\bar{B}_{\text{phys}}^0(t) \rightarrow f)), \\ &= \frac{1}{2} \int_{\text{bin}} ds_{12} ds_{23} |A|^2 e^{-\Gamma t} [(1 + |x|^2) \cosh(\Delta\Gamma t/2) + 2\text{Re}(\lambda) \sinh(\Delta\Gamma t/2)], \quad (4.9) \\ A_{CP}(t) &= \frac{\Gamma(B_{\text{phys}}^0(t) \rightarrow f) - \Gamma(\bar{B}_{\text{phys}}^0(t) \rightarrow f)}{\Gamma(B_{\text{phys}}^0(t) \rightarrow f) + \Gamma(\bar{B}_{\text{phys}}^0(t) \rightarrow f)}, \\ &= \frac{\int_{\text{bin}} ds_{12} ds_{23} |A|^2 [(1 - |x|^2) \cos(\Delta m t) - 2\text{Im}(\lambda) \sin(\Delta m t)]}{\int_{\text{bin}} ds_{12} ds_{23} |A|^2 [(1 + |x|^2) \cosh(\Delta\Gamma t/2) + 2\text{Re}(\lambda) \sinh(\Delta\Gamma t/2)]}. \quad (4.10)\end{aligned}$$

In $\Gamma(t)$, one does not distinguish $B_{\text{phys}}^0(t)$ and $\bar{B}_{\text{phys}}^0(t)$ decays, whereas one does in $A_{CP}(t)$. Thus, as usual, the measurement of the CP asymmetry requires tagging.

A comment should be made about eq. (4.10). The direct CP asymmetry compares $B^0 \rightarrow f$ and $\bar{B}^0 \rightarrow \bar{f}$ decays. Because $\bar{f} = f$ in two-body decays, there one refers to the coefficient of $\cos(\Delta m t)$ as the direct CP asymmetry. However, in three-body decays, because $\bar{f} \neq f$, the situation is different. Here the coefficient of $\cos(\Delta m t)$ compares $B^0 \rightarrow f$ and $\bar{B}^0 \rightarrow f$ decays, and so it is not actually a CP asymmetry.

In the above definitions there appear to be four observables, namely the coefficients of $\cos(\Delta m t)$, $\cosh(\Delta\Gamma t/2)$, $\sin(\Delta m t)$, and $\sinh(\Delta\Gamma t/2)$, as can be determined from $\Gamma(t)$ and the numerator of $A_{CP}(t)$. However, these coefficients are not all independent, as can be seen in the following identity:

$$\begin{aligned}|A|^2(1 + |x|^2) - |A|^2(1 - |x|^2) &= 2|A|^2|x|^2 = 2|A|^2|\lambda|^2, \\ &= 2|A|^2(\text{Re}(\lambda)^2 + \text{Im}(\lambda)^2). \quad (4.11)\end{aligned}$$

There are therefore only three independent observables.

One can perform a similar analysis using the last two equations of eq. (4.8). In this way one constructs the time-dependent CP-averaged rate and the CP asymmetry for the final state \bar{f} . There are again three independent observables. Thus, for a given $B_{d,s}^0 \rightarrow K_S h^+ h^-$ decay, there are a total of six observables: three each for the final states f and \bar{f} .

We now turn to the question of the number of unknown theoretical parameters, focusing on the decay pair $B_s^0 \rightarrow K_S \pi^+ \pi^-$ ($\bar{b} \rightarrow \bar{d}$) and $B_d^0 \rightarrow K_S K^+ K^-$ ($\bar{b} \rightarrow \bar{s}$). Consider first the $\bar{b} \rightarrow \bar{d}$ decay. The amplitudes for the various $B^0, \bar{B}^0 \rightarrow f, \bar{f}$ decays are given in eqs. (3.9), (3.10), (3.13) and (3.14). There are eight unknown parameters: $|T_d|$, $|P_d|$, $|\bar{T}_d|$,

$|\bar{P}_d|$, their three relative strong phases, and γ . With only six observables, γ cannot be extracted.

This can be remedied by also considering the U-spin conjugate $\bar{b} \rightarrow \bar{s}$ decay $B_d^0 \rightarrow K_S K^+ K^-$. Its $B^0, \bar{B}^0 \rightarrow f, \bar{f}$ amplitudes are also given in eqs. (3.9), (3.10), (3.13) and (3.14). Here too there are eight unknown parameters: $|T_s|$, $|P_s|$, $|\bar{T}_s|$, $|\bar{P}_s|$, their three relative strong phases, and γ . However, in the U-spin limit, we have $T_d = T_s \equiv T$ and $P_d = P_s \equiv P$. Thus, the two decays are described by the same eight unknown parameters. (As before, it is assumed that the B_d^0 - \bar{B}_d^0 and B_s^0 - \bar{B}_s^0 mixing phases are taken from independent measurements.) But there are now twelve observables, six for each of $B_s^0 \rightarrow K_S \pi^+ \pi^-$ and $B_d^0 \rightarrow K_S K^+ K^-$. On the other hand, it was noted in section 3.2 that there are two U-spin relation among the branching ratios and direct CP asymmetries of the $\bar{b} \rightarrow \bar{d}$ and $\bar{b} \rightarrow \bar{s}$ decays. Still, this leaves ten independent observables, which is more than the number of unknown parameters. Thus, assuming again that the magnitudes of the CKM matrix elements are known, γ can be extracted.

It must be mentioned that this method introduces a new systematic error. We have argued above that since the number of observables is greater than the number of unknowns, γ can be extracted. But this only works if all the observables are functions of the same unknowns. And because the measurements must be made using bins of the Dalitz plot, this does not hold exactly. Writing $A x = \langle f | \bar{B}^0 \rangle = \tilde{A}$, from eqs. (4.9) and (4.10) we have

$$\begin{aligned} BR &\propto \iint_{\text{bin}} ds_{12} ds_{23} (|A|^2 + |\tilde{A}|^2), \\ A_{\text{dir}}^{CP} &\propto \iint_{\text{bin}} ds_{12} ds_{23} (|A|^2 - |\tilde{A}|^2), \\ A_{\text{indir}}^{CP} &\propto \iint_{\text{bin}} ds_{12} ds_{23} \text{Im}[(q/p) A^* \tilde{A}]. \end{aligned} \quad (4.12)$$

If we define

$$\iint_{\text{bin}} ds_{12} ds_{23} |A|^2 \equiv |A'|^2, \quad \iint_{\text{bin}} ds_{12} ds_{23} |\tilde{A}|^2 \equiv |\tilde{A}'|^2, \quad (4.13)$$

we see that both BR and A_{dir}^{CP} are functions of A' and \tilde{A}' . However, A_{indir}^{CP} is not. We must make the approximation that

$$\iint_{\text{bin}} ds_{12} ds_{23} \text{Im}[(q/p) A^* \tilde{A}] \simeq \text{Im}[(q/p) A'^* \tilde{A}'], \quad (4.14)$$

and this introduces a systematic error. The above holds exactly for a single point of the Dalitz plot. Thus, the smaller the bins are, the better is the approximation, leading to a smaller systematic error. On the other hand, smaller bins lead to larger statistical errors. The bin size must therefore be chosen to minimize the total error.

5 U-spin breaking

As noted earlier, the method of combining measurements of decays related by U spin to extract γ was originally proposed in the context of two-body decays [9]. Here, there is a theoretical error due to unknown U-spin-breaking effects. This same difficulty arises when applying the method to three-body decays. In this section we examine the question of U-spin breaking as pertains to three-body decays.

The method described in the previous section for extracting γ involves combining measurements of pairs of three-body decays related by U-spin, such as $B_s^0 \rightarrow K_S \pi^+ \pi^-$ ($\bar{b} \rightarrow \bar{d}$) and $B_d^0 \rightarrow K_S K^+ K^-$ ($\bar{b} \rightarrow \bar{s}$). This method applies at a particular pair of Dalitz-plot points (bins). By repeating this analysis for all points, this provides multiple measurements of γ . These can then be averaged over the entire Dalitz plot, reducing the statistical error.

In the presence of U-spin breaking, the extracted value of γ , γ_{ext} , will differ from its true value, γ_{true} . Now, there are several different U-spin-breaking parameters. However, these parameters are all momentum dependent. Thus, their effect on the extracted value of γ will vary from point to point on the Dalitz plot. That is, if

$$\gamma_{\text{ext}} - \gamma_{\text{true}} = N, \quad (5.1)$$

it is likely that $N > 0$ at some points, and $N < 0$ at others. In this case, averaging over all Dalitz-plot points will also reduce the effect of U-spin breaking, so that $(\gamma_{\text{ext}})_{\text{avg}}$ will approach γ_{true} . If this occurs, the main theoretical error of the method will be significantly reduced.

Still, while this is a nice idea, how can we be certain that it is happening? Fortunately, there is a way of experimentally testing whether or not this behaviour is present in three-body decays. In eq. (3.12) it was shown that there is a relation among the observables of two decays related by U-spin reflection. Writing

$$-\frac{a_s^{CP}}{a_d^{CP}} \frac{\tau(B_d^0) b_s}{\tau(B_s^0) b_d} - 1 = n', \quad (5.2)$$

we have $n' = 0$ in the U-spin limit. By measuring $b_{d,s}$ and $a_{d,s}^{CP}$, and constructing the above ratio at each Dalitz-plot point, it is possible to experimentally determine if an average over all points leads to $n' \rightarrow 0$.

The above test requires a Dalitz analysis. A simpler test of U-spin breaking can be obtained by separately integrating the numerator and denominator of eq. (5.2) over the kinematically-allowed regions of the Dalitz plots (denoted by DP):

$$-\frac{\tau(B_d^0)}{\tau(B_s^0)} \frac{\int \int_{\text{DP}} ds_{12} ds_{23} a_s^{CP} b_s}{\int \int_{\text{DP}} ds_{12} ds_{23} a_d^{CP} b_d} - 1 = -\frac{A_s^{CP}}{A_d^{CP}} \frac{\tau(B_d^0) B_s}{\tau(B_s^0) B_d} - 1 = N'. \quad (5.3)$$

Unlike n' , which is defined using momentum-dependent quantities, N' depends only on integrated quantities, and hence does not depend on final-state momenta. Once again, we have $N' = 0$ in the U-spin limit.

Asymmetry ratio	U-spin prediction	LHCb result
$A^{CP}(B^+ \rightarrow \pi^+ K^+ K^-)/A^{CP}(B^+ \rightarrow K^+ \pi^+ \pi^-)$	-10.2 ± 1.5	-4.9 ± 2.0
$A^{CP}(B^+ \rightarrow \pi^+ \pi^+ \pi^-)/A^{CP}(B^+ \rightarrow K^+ K^+ K^-)$	-2.2 ± 0.2	-1.6 ± 0.5

Table 1. U-spin predictions for asymmetry ratios compared with LHCb measurements.

The above tests can be carried out using the measurements of $B_{d,s}^0 \rightarrow K_S h^+ h^-$ decays. However, it is not necessary to wait until these are made. Other pairs of three-body decays related by U spin are (i) $B^+ \rightarrow \pi^+ K^+ K^-$ ($\bar{b} \rightarrow \bar{d}$) and $B^+ \rightarrow K^+ \pi^+ \pi^-$ ($\bar{b} \rightarrow \bar{s}$), and (ii) $B^+ \rightarrow \pi^+ \pi^+ \pi^-$ ($\bar{b} \rightarrow \bar{d}$) and $B^+ \rightarrow K^+ K^+ K^-$ ($\bar{b} \rightarrow \bar{s}$). In ref. [33], group theory is used to write the factor n' of eq. (5.2) for these decay pairs in terms of U-spin-breaking parameters. These parameters take into account all U-spin-breaking effects, such as differences in the masses of the π and K mesons, differences in the properties of the resonances contributing to the decays (e.g., ρ , ϕ), etc. It is found that, to first order, n' is proportional to a linear combination of such parameters. That is, in the presence of U-spin breaking, $n' \neq 0$. However, the U-spin-breaking parameters are momentum-dependent. Using the same logic as before, it would not be surprising to find $n' > 0$ at some points and $n' < 0$ at others. If so, the average over all Dalitz-plot points will reduce the effect of U-spin breaking in the above relation.

These B^+ decays have recently been measured by LHCb [34, 35]. In ref. [30], the U-spin relation of eq. (5.3) is tested using data integrated over the Dalitz plot. We have updated these results with more recent data from refs. [36]. The updated results are shown in table 1. Unfortunately, at present the results are simply not precise enough to draw any conclusions. When the data improve, we will have a better idea of whether averaging (or integrating) over the Dalitz plot reduces the effect of U-spin breaking.

Finally, another source of U-spin breaking arises from the fact that π^\pm and K^\pm do not have the same mass, and similarly for B_d^0 and B_s^0 . This results in a difference between the kinematically-allowed phase space for a decay and that for its U-spin partner. Due to this difference, there will be regions of the Dalitz plots where the observables defined in section 4 can be obtained only for one of the two decays being compared. These regions must be excluded from the analysis, since our method for extracting γ works only for those regions of the Dalitz plots where the two decays have overlapping kinematically-allowed regions.

6 Conclusions

In 1999, R. Fleischer proposed a method for extracting γ using a pair of two-body decays whose amplitudes are related by U-spin symmetry ($d \leftrightarrow s$) [9]. It involves combining the measurements of the branching ratios and CP asymmetries, both direct and indirect, of the two decays. These decay amplitudes include penguin diagrams, which may receive important (loop-level) contributions from new physics. If so, the value of γ extracted using this method will disagree with its current value, which is obtained using tree-level decays.

In the present paper we adapt this method to charmless $B \rightarrow PPP$ decays (P is a pseudoscalar meson), specifically $B_{d,s}^0 \rightarrow K_S h^+ h^-$ ($h = K, \pi$). Time-dependent Dalitz analyses of the three-body decays can be used to measure the branching fractions and CP asymmetries. Note that it is not necessary to perform an isobar amplitude analysis of the Dalitz plot. We show that there are more observables than unknown theoretical parameters, so that γ can be extracted by fitting to the observables. The decay amplitudes for three-body decays depend on the momenta of the final-state particles. The method applies to each point of the Dalitz plot, and thus constitutes many independent measurements of γ .

The main source of theoretical error in the extraction of γ , which also applies to the method with two-body decays, is U-spin breaking. However, three-body decays offer the potential to reduce this error. The U-spin-breaking effects are also momentum-dependent. As such, the difference between the extracted value of γ and its true value may well vary, in both magnitude and sign, from point to point in the Dalitz plot. If this is the case, then averaging over the Dalitz plot will reduce the error due to U-spin breaking.

It is possible to test experimentally whether or not this behaviour is present in three-body decays. In the U-spin limit, there is a relation among the branching ratios and direct CP asymmetries of the two decays that are related by U spin. This applies to the decays $B^+ \rightarrow \pi^+ K^+ K^-$ ($\bar{b} \rightarrow \bar{d}$) and $B^+ \rightarrow K^+ \pi^+ \pi^-$ ($\bar{b} \rightarrow \bar{s}$), and $B^+ \rightarrow \pi^+ \pi^+ \pi^-$ ($\bar{b} \rightarrow \bar{d}$) and $B^+ \rightarrow K^+ K^+ K^-$ ($\bar{b} \rightarrow \bar{s}$), all of which have been measured. Unfortunately, the current data on these decays still has large errors, so that it is unclear whether U-spin breaking is small when averaged over the Dalitz plot. Future precision data in these channels will be able to clearly show the size of this U-spin breaking.

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